NEW RESULTS ON THE LOWER CENTRAL SERIES QUOTIENTS OF A FREE ASSOCIATIVE ALGEBRA

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Abstract. We continue the study of the lower central series and its associated graded components for a free associative algebra with n generators, as initiated in [FS]. We establish a linear bound on the degree of tensor field modules appearing in the Jordan-Hölder series of each graded component, which is conjecturally tight. We also bound the leading coefficient of the Hilbert polynomial of each graded component. As applications, we confirm conjectures of P. Etingof and B. Shoikhet concerning the structure of the third graded component.

1. Introduction and results

In this paper we consider the free associative algebra[1] $A := A_n$ on generators $x_1, \ldots, x_n$, for $n \geq 2$, and its lower central series filtration: $L_1 = A, L_{m+1} = [A, L_m]$. The corresponding associated graded Lie algebra is $B(A) = \oplus_m B_m(A)$, where $B_m(A) = L_m(A)/L_{m+1}(A)$. The natural grading on A by $S = (\mathbb{Z}_{\geq 0})^n$ descends to each B_m, and it is interesting to study the Hilbert series $h_{B_m}(t_1, \ldots, t_n)$ and $h_{B_m}(t) := h_{B_m}(t, \ldots, t)$. The formula for h_A is straightforward:

$$h_A(t_1 \ldots, t_n) = \frac{1}{1 - (t_1 + \cdots + t_n)},$$

and implies that $\dim A[d]$ grows exponentially in d. It is thus a somewhat surprising fact that for $m \geq 2$, $\dim B_m[d]$ grows as a polynomial in d of degree $n-1$. For $m = 2$, this was shown in [FS], and for $m \geq 3$ it was conjectured in [FS] and proven in [DE].

The proof of this fact is based on the representation theory of the Lie algebra W_n of polynomial vector fields on $\mathbb{C}^n$. Namely, in [FS], an action of W_n was constructed on each $B_m, m \geq 2$. It was conjectured there, and proved in [DE], that each B_m had a finite length Jordan-Hölder series, with respect to this action. The proof relied, firstly, on the observation in [FS] that all irreducible subquotients of B_m could be identified with certain tensor field modules $\mathcal{F}_\lambda$ associated to a Young diagram λ, and secondly, on a bound for the sizes $|\lambda|$ that could occur:

Theorem 1.1. [DE] *For $m \geq 3$ and $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_m(A_n)$, we have the following estimate on the size (i.e., the number of squares) of the Young diagram λ:*

$$|\lambda| \leq (m-1)^2 + 2\lfloor \frac{n-2}{2} \rfloor (m-1)$$

(where $\lfloor x \rfloor$ denotes the integer part of x).

[1] over $\mathbb{C}$, or any field of characteristic zero

This result, combined with well-known formulas for the Hilbert series of each $\mathcal{F}_\lambda$, established the finiteness of the Jordan-Hölder series, as well as the growth of $\dim B_m[d]$ as a degree $n-1$ polynomial in d, for $m \geq 2$. However, it was evident from experimental computations produced by Eric Rains that the maximal $|\lambda|$ for which $\mathcal{F}_\lambda$ occurs in B_m should grow linearly rather than quadratically in m. This indeed turns out to be the case. Namely, the main result of this paper is the following improvement of the bound of [DE]:

Theorem 1.2. *Let $m \geq 3$.*

(1) *For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_m(A_n)$,*
$$|\lambda| \leq 4m - 7 + 2\lfloor \frac{n-2}{2} \rfloor.$$

(2) *Let $n = 2$ or 3. For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_m(A_n)$,*
$$|\lambda| \leq 2m - 3$$

This is proven by means of some elementary commutative algebra, and a technical but very useful result:

Theorem 1.3. *Let $m \geq 2$.*

(1) *For all n, we have:*
$$B_2 = \sum_i [x_i, B_1].$$

(2) *For $n \geq 4$, we have:*
$$B_{m+1} = \sum_i [x_i, B_m] + \sum_{i \leq j} [x_i x_j, B_m] + \sum_{i<j<k} [x_i[x_j, x_k], B_m].$$

(3) *For $n = 3$, we have:*
$$B_{m+1} = \sum_i [x_i, B_m] + \sum_{i \leq j} [x_i x_j, B_m].$$

(4) *For $n = 2$, we have:*
$$B_{m+1} = \sum_i [x_i, B_m] + \sum_{i<j} [x_i x_j, B_m].$$

Remark 1.4. The presence of the cubic terms in (2) above appears superfluous from computer experiments, and we conjecture that they can be omitted (see also Lemma 5.2 and Corollary 5.3). Were this so, it would imply the bound $|\lambda| \leq 2m-3+2\lfloor \frac{n-2}{2} \rfloor$ in Theorem 1.2.

Definition 1.5. *The Hilbert polynomials $p_{mn}(d)$ are defined by*
$$p_{mn}(d) = \dim B_m(A_n)[d], \quad (d \gg 0).$$
The density*, a_{mn}, is the leading coefficient of p_{mn}, times $(n-1)!$.*

Example 1.6. The Hilbert polynomial of $\mathbb{C}[x_1, \ldots, x_n]$ is $\binom{n+d-1}{n-1}$, with leading coefficient $\frac{1}{(n-1)!}$, so the density is one. More generally, if $\lambda_1 \geq 2$ or $\lambda = (1^n)$, the density of $\mathcal{F}_\lambda$ is equal to the dimension of the irreducible representation V_λ of $\mathfrak{gl}_n$ with highest weight λ.

As a corollary to Theorem 1.3, we derive a bound for the density a_{mn}:

Corollary 1.7. *For $n = 2$, we have*

$$a_{m+1,n} \leq 3a_{m,n}.$$

For $n = 3$, we have

$$a_{m+1,n} \leq 9a_{m,n}.$$

For $n \geq 4$, we have

$$a_{m+1,n} \leq \frac{n^3 + 11n}{6} a_{m,n}.$$

As applications of the theory, we are able to prove the following conjecture of P. Etingof describing the complete structure of $B_3(A_n)$:

Theorem 1.8.

$$B_3(A_n) = \bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2, 1^{2i-1}, 0^{n-2i}).$$

As a corollary we derive the following conjecture of B. Shoikhet, which motivated the first conjecture:

Corollary 1.9. *Let $B_3(A_n)[1, \ldots, 1]$ denote the subspace of $B_3(A_n)$ of degree 1 in each generator. We have*

$$\dim B_3(A_n)[1, \ldots, 1] = (n-2)2^{n-2}.$$

Here, as elsewhere in the paper, we have used the abbreviation $(p_1, \ldots, p_n)$ instead of $\mathcal{F}_{(p_1,\ldots,p_n)}$. Combining our bounds in Theorem 1.2 with MAGMA[BCP] computations, we are also able to give the complete Jordan-Hölder series of $B_m(A_n)$ for many new m and n.

The structure of the paper is as follows. In Section 2 we briefly review the representation theory of the Lie algebra W_n, as well as the results of [FS] we will use. In Section 3 we prove Theorem 1.3 and Corollary 1.7. In Section 4 we prove Theorem 1.2. In Section 5, we prove Theorem 1.8, and present as a corollary a geometric description of the bracket map of $\bar{B}_1$ with B_2. In Section 6, we present the Jordan-Hölder series for $B_m(A_n)$ for small m and n.

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2. Preliminaries

In this section, we recall definitions for the Lie algebra W_n, the tensor field modules $\mathcal{F}_\lambda$, and the quantized algebra of even differential forms Ω_*^{ev}.

Definition 2.1. *Let $W_n = Der(\mathbb{C}[x_1, \ldots, x_n])$ denote the Lie algebra of polynomial vector fields,*

$$W_n = \oplus_i \mathbb{C}[x_1, \ldots, x_n]\partial_i,$$

with bracket $[p\partial_i, q\partial_j] = p\frac{\partial q}{\partial x_i}\partial_j - q\frac{\partial p}{\partial x_j}\partial_i$.

Let $\mathfrak{gl}_n$ denote the Lie algebra of n by n matrices. A Young diagram

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$$

with n rows gives rise to an finite dimensional irreducible representation V_λ of $\mathfrak{gl}_n$ contained in the space $(\mathbb{C}^{n*})^{\otimes|\lambda|}$ of covariant tensors of rank $|\lambda|$ on $\mathbb{C}^n$. Let $\widetilde{\mathcal{F}}_\lambda$ be the space of polynomial tensor fields of type V_λ on $\mathbb{C}^n$. As a vector space,

$$\widetilde{\mathcal{F}}_\lambda = \mathbb{C}[x_1, \ldots, x_n] \otimes V_\lambda.$$

It is well known that $\widetilde{\mathcal{F}}_\lambda$ is a representation of W_n with action given by the standard Lie derivative formula for action of vector fields on covariant tensor fields (see, e.g. [L] for details).

Theorem 2.2. [R] *If $\lambda_1 \geq 2$, or if $\lambda = (1^n)$, then $\widetilde{\mathcal{F}}_\lambda$ is irreducible. Otherwise, if $\lambda = (1^k, 0^{n-k})$, then $\widetilde{\mathcal{F}}_\lambda$ is the space $\Omega^k = \Omega^k(\mathbb{C}^n)$ of polynomial differential k-forms on $\mathbb{C}^n$, and it contains a unique irreducible submodule which is the space of all closed differential k-forms.*

Denote by $\mathcal{F}_\lambda$ the irreducible submodule of $\widetilde{\mathcal{F}}_\lambda$, so that $\mathcal{F}_\lambda = \widetilde{\mathcal{F}}_\lambda$ unless $\lambda = (1^k, 0^{n-k})$ for some $1 \leq k \leq n-1$.

Theorem 2.3. [R] *Any W_n-module on which the operators $x_i\partial_i$, for $i = 1, \ldots, n$, act semisimply with nonnegative integer eigenvalues and finite dimensional common eigenspaces has a Jordan-Hölder series whose composition factors are $\mathcal{F}_\lambda$, each occurring with finite multiplicity.*

As a trivial, but important, consequence of the definition of $\mathcal{F}_\lambda$, we have:

Proposition 2.4. *Suppose $\lambda \neq (1^k, 0^{n-k})$ for $1 \leq k \leq n-1$. Let h_{V_λ} denote the Hilbert series for V_λ. Then we have*

$$h_{\mathcal{F}_\lambda}(t_1, \ldots, t_n) = \frac{h_{V_\lambda}(t_1, \ldots, t_n)}{(1-t_1)\cdots(1-t_n)},$$

where we view V_λ as a $\mathbb{Z}^n$-graded vector space via its weight decomposition. In particular, we observe that $h_{\mathcal{F}_\lambda}(t_1, \ldots, t_n)(1-t_1)\cdots(1-t_n)$ is a polynomial of total degree $|\lambda|$.

Let Ω^{ev} be the space of polynomial differential forms on $\mathbb{C}^n$ of even rank, and Ω^{ev}_{ex} its subspace of even exact forms. These spaces are graded by setting $\deg(x_i) = \deg(dx_i) = 1$. Recall that Ω^{ev} is a *commutative* algebra with respect to the wedge product.

Definition 2.5. *The multiplication $a * b = a \wedge b + da \wedge db$ defines an associative product on Ω^{ev}. By Ω^{ev}_* we will mean the space Ω^{ev} equipped with the $*$-product, and call it the quantized algebra of even differential forms.*

Definition 2.6. *Let Z denote the image of $A[A,[A,A]]$ in B_1, which was shown in* [FS] *to be central in B. We define $\bar{B}_1$ to be the quotient, $\bar{B}_1 = B_1/Z$, and define $\bar{B} = \bar{B}_1 \ \oplus \ (\oplus_{i\geq 2} B_i)$.*

Clearly $\bar{B}$ inherits the grading from B, and is thus a graded Lie algebra generated in degree 1.

Theorem 2.7. [FS] *There is a unique isomorphism of algebras,*

$$\xi : \Omega_*^{ev} \to A/A[A,[A,A]],$$
$$x_i \mapsto x_i,$$

which restricts to an isomorphism $\xi : \Omega_{*,ex}^{ev} \xrightarrow{\sim} B_2$*, and descends to an isomorphism* $\xi : \Omega_*^{ev}/\Omega_{*,ex}^{ev} \xrightarrow{\sim} \bar{B}_1$*.*

Theorem 2.8. [FS] *The action of* W_n *on* $\bar{B}_1 \cong \Omega^{ev}/\Omega_{ex}^{ev}$ *by Lie derivatives uniquely extends to an action of* W_n *on* $\bar{B}$ *by grading-preserving derivations.*

Thus, for each $m \geq 2$, B_m is a W_n-module clearly satisfying the conditions of Theorem 2.3, and thus has composition factors $\mathcal{F}_\lambda$, each occurring with finite multiplicity.

3. Proof of Theorem 1.3 and Corollary 1.7

Lemma 3.1. *We have the following identity, which may be directly checked.*

$$\begin{aligned}[u^3,[v,w]] =& 3[u^2,[uv,w]] - 3[u,[u^2v,w]] + \frac{3}{2}[u^2,[v,[u,w]]] \\ &- \frac{3}{2}[u,[v,[u^2,w]]] + [u,[u,[u,[v,w]]]] \\ &- \frac{3}{2}[u,[u,[v,[u,w]]]] + \frac{3}{2}[u,[v,[u,[u,w]]]].\end{aligned}$$

Corollary 3.2. *Let* $S(a,b,c)$ *be the symmetrized sum*

$$S(a,b,c) = \frac{1}{6}(abc + bca + cab + acb + cba + bac).$$

Then, for all a, b, c*, we have that*

$$[S(a,b,c), B_m] \subset [ab, B_m] + [bc, B_m] + [ca, B_m] + [a, B_m] + [b, B_m] + [c, B_m] \subset B_{m+1}.$$

Proof. In Lemma 3.1, set $u = t_1a + t_2b + t_3c$, v equal to any element of A_n, w equal to any element of B_{m-1}, and take the coefficient of $t_1t_2t_3$. The result follows. □

Lemma 3.3. *Let* $E \subset \Omega_*^{ev}$ *be the span of* $S(a,b,c)$*, where* a, b*, and* c *are of positive degree* ($\deg(x_i) = \deg(dx_i) = 1$)*. Let* X *be the span of 1,* x_i*,* x_ix_j *for* $i \leq j$*,* $x_idx_j \wedge dx_k$ *for* $i < j < k$*. Then* $\Omega_*^{ev} = X + E + \Omega_{ex,*}^{ev}$*.*

Proof. Ω_*^{ev} has a finite length descending filtration by rank of forms, such that the associated graded is the usual commutative algebra of even forms. Therefore, it is sufficient to check the same statement for the commutative algebra of even forms. Then Ω^{ev}/E is spanned by

$$1,\ x_i,\ dx_i \wedge dx_j,\ x_ix_j,\ x_idx_j \wedge dx_k,\ dx_i \wedge dx_j \wedge dx_k \wedge dx_l.$$

As the forms

$$dx_i \wedge dx_j,\ x_idx_i \wedge dx_j,\ dx_i \wedge dx_j \wedge dx_k \wedge dx_l$$

are exact, the Lemma follows. □

Now we proceed to prove the theorem. (1) Follows from the isomorphism $B_2 \cong \Omega_{ex}^{ev}$. For (2), we have: $B_{m+1} = [\xi(\Omega^{ev}), B_m]$. Now, by Lemma 3.3 and Corollary 3.2, $[\xi(\Omega^{ev}), B_m] = [\xi(X), B_m]$, so the statement follows.

For (4), we observe the following identity, which may be checked directly:

Lemma 3.4.

$$\begin{aligned}[x^2,[y,w]] =&2[x,[xy,w]]+[y,[x^2,w]]-2[xy,[x,w]]-[w,[x,[y,x]]].\\ 6[x^2,[xy,w]] =&12[x,[x^2y,w]]+4[y,[x^3,w]]-6[xy,[x^2,w]]\\ &-3[x^2,[y,[x,w]]]-3[y,[x^2,[x,w]]]+9[x,[y,[x^2,w]]]\\ &-3[x,[x,[x,[y,w]]]]+3[x,[x,[y,[x,w]]]]-3[x,[y,[x,[x,w]]]]\\ &-[y,[x,[x,[x,w]]]].\end{aligned}$$

Setting $x=x_1$, $y=x_2$, and letting $w \in B_{m-2}$, we get that $[x_i^2, B_{m-1}] \subset [x_1, B_{m-1}] + [x_2, B_{m-1}] + [x_1x_2, B_{m-1}]$ as desired.

For (3), we use a lemma:

Lemma 3.5. [DE] *There exists a function $\epsilon : S_m \to \mathbb{Q}$, such that*

$$[a_0,[a_1,[\cdots[a_{m-1},a_m]\cdots]=\sum_{\sigma\in S_m}\epsilon(\sigma)[a_{\sigma(1)},[a_{\sigma(2)},[\cdots[a_{\sigma(m)},a_0]\cdots].$$

Thus an element $x=[a_0,[a_1,[\cdots[a_{m-1},a_m]\cdots]$, with $a_0 \in \Omega^2$ is a sum of elements with a_0 in the innermost bracket. The innermost bracket will then be in $\zeta(\Omega^4)$, which is zero in B_2 for $n=3$. □

Corollary 1.7 follows immediately by counting the dimension of X.

4. Proof of Theorem 1.2

Let $\xi : \Omega^{ev} \to \bar{B}_1$ be the Feigin-Shoikhet surjective map from Theorem 2.7. Consider the map $f_m : (\Omega^{ev})^{\otimes m} \to B_m$ given by

$$f_m(a_1,\ldots,a_m) = [\xi(a_1),[\xi(a_2),\ldots[\xi(a_{m-1}),\xi(a_m)]].$$

Since $\bar{B}$ is generated in degree 1, this map is surjective. By Theorem 1.3, the restriction of f_m to $Y := (\Omega^0)^{\otimes m}$ is surjective for $n=2,3$, while the restriction to $Y := (\Omega^0+\Omega^2)^{m-2} \otimes (\oplus_{j+k\le\lfloor\frac{n-2}{2}\rfloor}\Omega^{2j}\otimes\Omega^{2k})$ is surjective for $n \ge 4$. The idea of the proof will be to find a large W_n-submodule K of Y such that $f_m|_K = 0$ and, such that the composition factors $\mathcal{F}_\lambda$ of Y/K satisfy the bound on $|\lambda|$ given in the theorem. This will obviously imply Theorem 1.2.

Lemma 4.1. *Let $K \subset Y$ be the submodule spanned by elements of the form:*

(1) $p_1 \otimes \cdots \otimes p_{m-3-i} \otimes (1\otimes b) * (a\otimes 1 - 1\otimes a)^3 \otimes p_{m-2-i} \otimes \cdots \otimes p_{m-2}$, *for* $0 \le i \le m-3$, *where* $a \in \Omega^0_*, b \in \Omega^{ev}_*$, *and* $p_i \in \Omega^{ev}_*$, and
(2) $p_1 \otimes \cdots \otimes p_{m-2} \otimes (b_1\otimes b_2) * (a\otimes 1 - 1\otimes a)^2$, *where* $a \in \Omega^0_*, b_1, b_2 \in \Omega^{ev}_*$, *and* $p_i \in \Omega^{ev}_*$.

Then $f_m|_K = 0$.

Proof. Elements of type 1 are killed by f_m by Lemma 3.1, as

$$(1\otimes b)*(a\otimes 1-1\otimes a)^3 = (a^3\otimes b - 3a^2\otimes b*a + 3a\otimes b*a^2 - 1\otimes b*a^3).$$

Elements of type 2 are killed, as, for any functions a, b_1, b_2,

$$d(b_1a^2)\wedge db_2 - 2d(b_1a)\wedge d(b_2a) + db_1\wedge d(b_2a^2) = 0.$$

□

Let $K_0 \subset Y$ be the associated graded of K under the "rank of forms" filtration, and let K_0' be the span of elements of type 1 and 2 in the commutative algebra of forms. The Jordan-Hölder series of Y/K is the same as the Jordan-Hölder series of Y/K_0, so it is dominated by the Jordan-Hölder series of Y/K_0'. So, it suffices to show that all $\mathcal{F}_\lambda$ occurring in Y/K_0' satisfy the bound of the theorem. We will do this by precisely computing the Hilbert series of Y/K_0'. First, we recall a well-known fact from commutative algebra.

Lemma 4.2. *Let B be a commutative algebra, and $I \subset B \otimes B$ be the kernel of the multiplication homomorphism $\mu : B \otimes B \to B$. Also, let $k \in \mathbb{N}$. Then, I^k is spanned by elements of the form $(1 \otimes b)(a \otimes 1 - 1 \otimes a)^k$, where $a, b \in B$.*

Proof. Obviously $(a \otimes 1 - 1 \otimes a) \in I$, so $(1 \otimes b)(a \otimes 1 - 1 \otimes a)^k \in I^k$. Now I^k is generated by $(a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_k \otimes 1 - 1 \otimes a_k)$, for $a_i \in B$. Because for every vector space V, $S^k(V)$ is spanned by $\{v^{\otimes k} | v \in V\}$, such elements can be obtained as linear combinations of elements of the form $(a \otimes 1 - 1 \otimes a)^k$. So, I^k is spanned by $(b_1 \otimes b_2)(a \otimes 1 - 1 \otimes a)^k$. But

$$\begin{aligned}(b_1 \otimes b_2)(a \otimes 1 - 1 \otimes a)^k =& (1 \otimes b_2)(b_1 a \otimes 1 - 1 \otimes b_1 a)(a \otimes 1 - 1 \otimes a)^{k-1} \\ &- (1 \otimes a b_2)(b_1 \otimes 1 - 1 \otimes b_1)(a \otimes 1 - 1 \otimes a)^{k-1},\end{aligned}$$

so we are done. □

We let $R = \mathbb{C}[x_1, \dots, x_n]^{\otimes m} = \mathbb{C}[x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2, \dots, x_1^m, \dots, x_n^m]$. Let J_j, for $1 \le j \le m-1$ be the ideal in R generated by $X_i^j := x_i^j - x_i^{j+1}$. Let

$$J = \sum_{j=1}^{m-2} J_j^3 + J_{m-1}^2.$$

Corollary 4.3. $K_0' = JY$, *so* $Y/K_0' = Y/JY$.

Proof. This follows immediately from Lemma 4.1 and Lemma 4.2. □

Now, we can finish the proof of the theorem. Namely, Y is a free module over R, so $h_{Y/JY} = h_{R/J} \cdot h_X$, where h_X is the Hilbert series of the generators over R of Y. For $n = 2, 3$, we have $h_X = 1$, while for $n \ge 4$, we have:

$$h_X = (1 + \sigma_2)^{m-2} \cdot \sum_{j+k \le 2\lfloor \frac{n-2}{2} \rfloor} \sigma_{2j} \cdot \sigma_{2k}. \tag{1}$$

where $\sigma_l = \sum_{i_1 < \dots < i_l} t_{i_1} \cdots t_{i_l}$ are elementary symmetric functions. Now, from the description of J, we compute

$$h_{R/J} = \frac{(1 + \sum t_i + \sum_{i \le j} t_i t_j)^{m-2}(1 + \sum t_i)}{(1 - t_1) \cdots (1 - t_n)}.$$

Thus $h_{Y/JY} \cdot (1 - t_1) \cdots (1 - t_n)$ is a polynomial of degree less than or equal to $2m - 3$, for $n = 2, 3$ and $2m - 3 + 2(m - 2) + 2\lfloor \frac{n-2}{2} \rfloor$ for $n \ge 4$, proving Theorem 1.2.

Remark 4.4. For n=2, it follows from the proof of the theorem that the image, $f_m(v_m)$, of

$$v_m = (x_1 - x_2)(y_1 - y_2)\cdots(x_{m-2} - x_{m-1})(y_{m-2} - y_{m-1})(x_{m-1} - x_m)$$

generates a copy of $(m-1, m-2)$, if it is non-zero (here we have used the notation $x_k := x_1^k, y_k := x_2^k$). We conjecture that $f_m(v_m)$ is indeed non-zero, and that $(m-1, m-2)$ occurs with multiplicity one in $B_m(A_2)$. Furthermore we conjecture that for (p, q) occurring in $B_m(A_2)$, one has $p \leq m-1$. For $m \leq 7$, these conjectures are confirmed in Theorem 6.1.

5. The complete structure of $B_3(A_n)$

In this section, we prove Theorem 1.8, which was first conjectured by P. Etingof. Firstly, we show that only representations of the form $(2, 1^{2i-1}, 0^{n-2i})$ can appear in the Jordan-Hölder series of $B_3(A_n)$; this is accomplished by a strengthening of Theorem 1.2 in the case $m = 3$. Secondly, we use the representation theory of $\mathfrak{gl}_n$ to bound the multiplicity of each $(2, 1^{2i-1}, 0^{n-2i})$ in $B_3(A_n)$ to at most one. Thirdly, we exhibit a non-zero vector in each $(2, 1^{2i-1}, 0^{n-2i})$ by representing the generators of A_n in a certain quotient algebra. Finally we show that the sum appearing in the Theorem is direct. Steps 2-4 were explained to us by P. Etingof.

5.1. **Step one.**

Lemma 5.1. *Let $m \geq 3$. Then no $\mathcal{F}_{(1^k, 0^{n-k})}$ occurs in B_m.*

Proof. **Case 1: $k = n$.**
As a representation of S_n, the polylinear part (i.e., the part of degree 1 in each variable) of $(1, \ldots, 1)$ is isomorphic to the sign representation. On the other hand, the polylinear part of A_n (i.e., the span of monomials of the form $x_{\sigma(1)} \cdots x_{\sigma(n)}$ for $\sigma \in S_n$) is clearly a copy of the regular representation, which contains the sign representation exactly once, and thus the total multiplicity of $(1, \ldots, 1)$ in all $B_m(A_n)$ is equal to 1. In [FS], it is shown that when n is odd, $(1, \ldots, 1) \subset \bar{B}_1(A_n)$ and when n is even, $(1, \ldots, 1) \subset B_2(A_n)$, so $(1, \ldots, 1)$ cannot occur in B_m for $m \geq 3$.
Case 2: $k < n$.
Suppose that $(1^k, 0^{n-k})$ occurs in $B_m(A_n)$. If $k < n$, then this means $B_m(A_k)$ contains a copy of (1^k), which contradicts Case 1. $\square$

Lemma 5.2. *We have:*

$$[x[y, z], [w, v]] = [x, [w[y, z], v]] - [y, [w[x, z], v]] + [z, [w[x, y], v]] \mod L_4$$

Proof. Let G denote the symmetric group on the set $\{x, y, z, w, v\}$, and let ψ denote the RHS minus the LHS of the identity. Applying the Jacobi identity to the term $[x[y, z], [w, v]]$, we have

$$\psi = [x, [w[y, z], v]] - [y, [w[x, z], v]] + [z, [w[x, y], v]] - [w, [x[y, z], v]] + [v, [x[y, z], w]].$$

Recall the isomorphism $\xi : \Omega_{ex}^{ev, \geq 2} \xrightarrow{\sim} B_2$ of Theorem 2.7; for any $\alpha, \beta, \gamma, \delta \in \bar{B}_1$, we have $[\alpha[\beta, \gamma], \delta] = 4\xi(d\alpha \wedge d\beta \wedge d\gamma \wedge d\delta) \mod L_3$. Thus we may re-express ψ:

$$\begin{aligned}\psi =& 4\big([x, \xi(dw \wedge dy \wedge dz \wedge dv)] - [y, \xi(dw \wedge dx \wedge dz \wedge dv)] + [z, \xi(dw \wedge dx \wedge dy \wedge dv)] \\ &- [w, \xi(dx \wedge dy \wedge dz \wedge dv)] + [v, \xi(dx \wedge dy \wedge dz \wedge dw)]\big).\end{aligned}$$

We see immediately that ψ is skew symmetric with respect to each of the permutations (x, y), (y, z), (z, w) and (w, v). As these permutations generate G, it follows that if ψ is non-zero, then $\mathbb{C}\psi$ is isomorphic to the sign representation. However, we have already seen in the proof of Lemma 5.1 that the sign representation does not occur in the polylinear part of $B_m(A_n)$ for any $m \geq 3$, and thus $\psi = 0 \mod L_4$. □

Corollary 5.3. $B_3(A_n) = \sum_i [x_i, B_2] + \sum_{i\leq j}[x_i x_j, B_2]$.

Proof. Lemma 5.2, combined with Corollary 3.2, allows us to reduce the degree of any expression in the outer slot of degree three or greater, thus leaving only the quadratic terms, as desired. □

Lemma 5.4. *For $\mathcal{F}_\lambda$ appearing in the Jordan-Hölder series of $B_3(A_n)$, we have*

$$|\lambda| \leq 3 + 2\lfloor \frac{n-2}{2} \rfloor = \begin{cases} n, & n \text{ odd} \\ n+1, & n \text{ even} \end{cases}$$

Proof. By Corollary 5.3, the map f_3 in the proof of Theorem 1.2 is surjective when restricted to $Y := (\Omega^0) \otimes (\oplus_{j+k\leq \lfloor \frac{n-2}{2} \rfloor} \Omega^{2j} \otimes \Omega^{2k})$. Thus we may omit the factor $(1 + \sigma_2)$ in equation (1), and we compute that $h_{Y/JY} \cdot (1 - t_1) \cdots (1 - t_n)$ is a polynomial of degree less than or equal to $3 + 2\lfloor \frac{n-2}{2} \rfloor$. □

Corollary 5.5. *If $\mathcal{F}_\lambda$ appears in the Jordan-Hölder series of $B_3(A_n)$, then $\lambda = (2, 1^{2i-1}, 0^{n-2i})$ for some $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. We proceed by induction on n, the case $n = 2$ having been proved in [DKM]. First, suppose for contradiction that some $\mathcal{F}_\lambda$ occurs in the Jordan-Hölder series for $B_3(A_n)$, with $\lambda_1 \geq 3$. Then we have $\lambda_n = 0$ by Lemma 5.4, which implies that $(\lambda_1, \ldots, \lambda_{n-1})$ occurs in $B_3(A_{n-1})$. This contradicts the induction assumption. Thus $\lambda_1 \leq 2$.

Let us again suppose for contradiction that some $\mathcal{F}_\lambda$ occurs with $\lambda_1 = \lambda_2 = 2$. Then $\lambda_n = 0$, and so $(2, 2, \lambda_3, \ldots, \lambda_{n-1})$ occurs in $B_3(A_{n-1})$, which contradicts the induction assumption.

Furthermore, by Lemma 5.1, no factors $(1^k, 0^{n-k})$ may occur in B_3. The only remaining possibilities are of the form $(2, 1^k, 0^{n-k-1})$,and it remains only to show that k must be odd. Indeed, if k is even, say $\lambda = (2, 1^k, 0^{n-k-1})$, and $\mathcal{F}_\lambda$ occurs in $B_3(A_n)$, then $(2, 1^k)$ occurs in $B_3(A_{k+1})$, which contradicts Lemma 5.4. □

5.2. **Step two.**

Lemma 5.6. *The multiplicity of $(2, 1^{n-1})$ in $B_3(A_n)$ is at most one.*

Proof. Let $n = 2k$. By Lemma 5.3, we have a surjection:

$$f_3 : \sum_{p=0}^{k-1} Y_p \to B_3,$$

where $Y_p = \Omega_0 \otimes \Omega_0 \otimes \Omega_{2p}$. As in the proof of Theorem 1.2, we let

$$R = \mathbb{C}[x_1^1, \ldots, x_n^1, x_1^2, \ldots x_n^2, x_1^3, \ldots, x_n^3].$$

We can identify Y_p with the free module over R with generators

$$X_p = \{dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_{2p}}\},$$

and Hilbert series $h_{X_p} = \sigma_{2p}$. We let J_j, for $j = 1, 2$, denote the ideal generated by $X_i^j := x_i^j - x_i^{j+1}, i = 1, \ldots n$, and let $J = J_1^3 + J_2^2$. Then, Lemmas 4.1 and 4.2

imply that JY_p is in the kernel of f_3. As J is W_n invariant, we have a surjection of W_n modules $f_3 : Y_p/JY_p \to f_3(Y_p) \subset B_3$. We compute:

$$h_{Y_p/JY_p} = h_{R/J} h_{X_p} = \frac{(1 + \sum t_i + \sum_{i \leq j} t_i t_j)(1 + \sum t_i)\sigma_{2p}}{(1-t_1)\cdots(1-t_n)}.$$

Thus $h_{Y_p/JY_p} \cdot (1-t_1)\cdots(1-t_n)$ is a polynomial of degree less than or equal to $2p+3$, so that the maximal size for λ which can appear in each Y_p/JY_p is $2p+3$. Thus only Y_{k-1} can contribute to multiplicity of $(2, 1^{n-1})$.

So we consider the image $f_3(Y_{k-1}) \subset B_3$. Scalars in the outer slot are sent to zero, and we can view the inner two slots naturally as the degree n subspace of $B_2 \cong \Omega_{ex}^{ev, \geq 2}$. We can thus write $f_3(Y_{k-1})$ as a quotient of

$$M = (\oplus_{j \geq 1} S^j V) \otimes \Lambda^n V \otimes (\oplus_{m \geq 0} S^m V).$$

The multiplicity of the $\mathfrak{gl}_n$ module $(2, 1^{n-1})$ in M is equal to the multiplicity of $(1, 0^{n-1})$ in $(\Lambda^n V)^* \otimes M = (\oplus_{j\geq 1} S^j V) \otimes (\oplus_{m \geq 0} S^m V)$, which is clearly one. $\square$

Corollary 5.7. *A cyclic generator of* $(2, 1^{n-1})$ *in* $B_3(A_n)$ *is*

$$v_n = [x_1, [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]].$$

Proof. Apply f_3 to the generator $x_1 \otimes x_1 \wedge \cdots \wedge x_n$ of $(2, 1^{n-1})$ in M. $\square$

Corollary 5.8. *The multiplicity of* $(2, 1^{2i-1}, 0^{n-2i})$ *in* $B_3(A_n)$ *is at most one.*

Proof. If we assume to the contrary that $(2, 1^{2i-1}, 0^{n-2i})$ appears with multiplicity greater than one, then it follows that $(2, 1^{2i-1})$ occurs in $B_3(A_{2i})$ with multiplicity greater than one, contradicting Lemma 5.6. $\square$

5.3. **Step three.** Fix some $k \in \mathbb{N}$, let A be any algebra, let E denote the exterior algebra in generators $\zeta_1, \ldots \zeta_k$, and let $B = A \otimes E$. We denote by E^i, E_+, E_-, and $E_+^{\geq j}$, and $E_-^{\geq j}$ the i-th graded component, even part, odd part, and the even and odd parts of degree at least j, respectively.

Lemma 5.9. *We have*

$$\begin{aligned}
[B,B] =& [A,A] \otimes (E^0 \oplus E_-) \oplus A \otimes E_+^{\geq 2}.\\
[B,[B,B]] =& [A,[A,A]] \otimes (E^0 \oplus E^1) \oplus A[A,A] \otimes E_+^{\geq 2} \oplus [A,A] \otimes E_-^{\geq 3}.\\
[B,[B,[B,B]]] =& [A,[A,[A,A]] \otimes (E^0 \oplus E^1) \oplus ([A, A[A,A]] + A[A,[A,A]]) \otimes E^2\\
& \oplus [A, A[A,A]] \otimes E_-^{\geq 3} \oplus A[A,A] \otimes E_+^{\geq 4}.
\end{aligned}$$

Proof. A direct computation using the skew commutativity of E. $\square$

Corollary 5.10. *We have*

$$\begin{aligned}
[B,[B,B]]/[B,[B,[B,B]]] =& ([A,[A,A]]/[A,[A,[A,A]]]) \otimes (E^0 \oplus E^1)\\
& \oplus (A[A,A]/([A,A[A,A]] + A[A,[A,A]])) \otimes E^2\\
& \oplus ([A,A]/[A,A[A,A]]) \otimes E_-^{\geq 3}.
\end{aligned}$$

Proposition 5.11. *The generator* $v_n = [x_1, [x_1 x_2] \cdots [x_{2k-1}, x_{2k}]]$ *of* $(2, 1^{n-1})$ *is nonzero in* $B_3(A_n)$.

Proof. Clearly it suffices to find some algebra C and elements $x_1, \dots x_{2k}$ such that the expression defining v_n is not in $L_4(C)$. We let A be the free algebra in two generators a, b, let E be the exterior algebra in generators $\zeta_0, \dots, \zeta_{2k}$,and let $B = A \otimes E$. We set $x_i = \zeta_i$ for $i = 2, \dots, 2k$, and $x_1 = a\zeta_0 + b\zeta_1$. Then a direct computation shows:

$$[x_1, [x_1 x_2] \cdots [x_{2k-1} x_{2k}]] = 2^{k+1}[ab] \otimes \zeta_0 \wedge \cdots \wedge \zeta_{2k}.$$

By the corollary, this is nonzero in $[B, [B, B]]/[B, [B, [B, B]]]$, as its component in $([A, A]/[A, A[A, A]]) \otimes E^{\geq 3}_{-}$ is non-zero. The proposition is proved. □

Corollary 5.12. *The Jordan-Hölder series of* $B_3(A_n)$ *is* $\{(2, 1^{2i-1}, 0^{n-2i})\}_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor}$, *each appearing with multiplicity one.*

5.4. **Step four.**

Proposition 5.13. *Each* $(2, 1^{2i-1}, 0^{n-2i})$ *is a submodule, so the sum in Theorem 1.8 is direct.*

Proof. Let $v_k = [x_1, [x_1 x_2] \cdots [x_{2k-1} x_{2k}]] \in B_3(A_n)$, and let X_k be the submodule generated by v_k. Clearly, $\partial_i v_k = 0$ for all i, so the JH series of X_k involves only terms of the form $(2, 1^{2r-1}, 0^{n-2k})$, where $r \geq k$. On the other hand, we saw in the proof of Lemma 5.6 that this representation cannot involve $\mathcal{F}_\lambda$ with more than $2k+1$ boxes. Thus, we must have $X_k = (2, 1^{2k-1}, 0^{n-2k})$ as desired. □

Corollary 1.9 can now be derived by counting the dimension of the graded component for the decomposition in Theorem 1.8.

5.5. **A geometric description of the bracket of $\bar{B}_1$ and B_2.** The isomorphism in Theorem 1.8 allows us the following geometric description of the bracket map,

$$[-,-] :(\bar{B}_1/\mathbb{C}) \otimes B_2 \to B_3.$$
$$a \otimes b \mapsto [a, b].$$

To begin, we identify $\bar{B}_1/\mathbb{C} \cong \Omega^{odd}_{ex}$, and $B_2 \cong \Omega^{ev, \geq 2}_{ex}$, as in [FS]. Also, we identify B_3 with the direct sum of Theorem 1.8, by sending $[x_1, [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]]$ to $\pi(dx_1 \otimes dx_1 \wedge \cdots \wedge dx_{2k})$, where $\pi : V \otimes \Lambda^{2k} V \to (2, 1^{2k-1}, 0^{n-2k})$ is the standard projection.[2]

Definition 5.14. *Let* $\psi_s : \Lambda^{s+1} V \otimes \Lambda^q V \to (2, 1^{s+q-1}, 0, \dots, 0)$ *be the unique (up to scaling) morphism of* $\mathfrak{gl}_n$*-modules:*

$$\psi_s(v_0 \wedge \cdots \wedge v_s \otimes b) = \sum_{i=0}^{s} (-1)^i \pi(v_i \otimes (v_0 \wedge \cdots \hat{v}_i \cdots \wedge v_s \wedge b))$$

Proposition 5.15. *For* $a \in \Omega^{2p+1}_{ex}, b \in \Omega^{ev, \geq 2}_{ex}$, *the bracket map is given by the formula:*

$$[a, b] = \psi_{2p}(a \otimes b).$$

Proof. This follows immediately from Lemma 5.1 by induction on p. □

In particular, the proposition implies that the bracket map is induced from a fiberwise morphism of the corresponding vector bundles on $\mathbb{A}^n$.

[2] Here, to fix normalizations unambiguously, we define the $\mathfrak{gl}(V)$-modules $(2, 1^{p-1}, 0, \dots, 0)$ as the unique submodules of the corresponding type in $V \otimes \Lambda^p V$.

6. Decompositions

Using the computational algebra system MAGMA [BCP], we were able to produce the bigraded Hilbert series of $B_m(A_2)$ up to degree 12, and the tri-graded Hilbert series for $B_m(A_3)$ up to degree 8. Combined with Theorem 1.2, these results imply the following:

Theorem 6.1. *The Jordan-Hölder series of $B_m(A_2)$ for $m = 2, \dots, 7$ are*

$$\begin{aligned}
B_2(A_2) =& (1,1)[\mathrm{FS}] \\
B_3(A_2) =& (2,1)[\mathrm{DKM}] \\
B_4(A_2) =& (3,1) + (3,2)[\mathrm{DKM}] \\
B_5(A_2) =& (4,1) + (3,2) + (4,2) + (4,3) \\
B_6(A_2) =& (5,1) + (4,2) + (3,3) + 2(5,2) + 2(4,3) + (5,3) + (5,4) \\
B_7(A_2) =& (6,1) + 2(5,2) + 2(4,3) + 2(6,2) + 3(5,3) + 2(4,4) + 2(6,3) + 2(5,4) \\
& + (6,4) + (6,5)
\end{aligned}$$

The Jordan-Hölder series of $B_m(A_3)$ for $m = 2, \dots, 5$ are

$$\begin{aligned}
B_2(A_3) &= (1,1,0)[\mathrm{FS}] \\
B_3(A_3) &= (2,1,0)[\mathrm{DE}] \\
B_4(A_3) &= (3,1,0) + (2,1,1) + (3,2,0) + (2,2,1) \quad (\text{conj. in [FS]}) \\
B_5(A_3) &= (4,1,0) + (3,2,0) + (3,1,1) + (2,2,1) + (4,2,0) + (4,1,1) \\
&\quad + 3(3,2,1) + (2,2,2) + (4,3,0) + (3,3,1)
\end{aligned}$$

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